A CONSEQUENCE OF LITTLEWOOD'S CONDITIONAL ESTIMATES FOR THE RIEMANN ZETA-FUNCTION

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Assuming the Riemann hypothesis (RH) and using Littlewood's conditional estimates for the Riemann zeta-function, we provide an estimate related to an approach of Y. Motohashi to the zero-free region.

Key words: Riemann zeta-function, Riemann hypothesis.

1. Introduction. The approach of Y. Motohashi [1] to the zero-free region of the Riemann zeta-function extended by the author in [2] may be modified to give regions free of large values of some products, which contain finite products $\prod_j \zeta(s_j)$. On the Riemann hypothesis, one can obtain upper bounds for such products for $s_j = 1 + t_j$ using the method of Littlewood. To prove our result on regions free of large values we also use an Ω-theorem for $\prod_j \frac{1}{\zeta(s_j)}$, where $s_j = \sigma_j + i(t_j + h_j)$ with h_j lying in short intervals around t_j and $\sigma_j \geqslant 1$. The Ω-theorem depends on a version of Kronecker's theorem with an explicit upper bound.

2. Lemmas.

Lemma 1. *On the Riemann hypothesis, uniformly for $\frac{1}{2} < \sigma_0 \leqslant \sigma \leqslant \frac{9}{8}$ and $t \geqslant e^{27}$ we have*

$$\log \zeta(s) \ll \begin{cases} \log \frac{1}{\sigma-1} & \text{if} \quad 1 + \frac{1}{\log\log t} \leqslant \sigma \leqslant \frac{9}{8}, \\ \frac{(\log t)^{2-2\sigma}-1}{(1-\sigma)\log\log t} + \log\log\log t & \text{if} \quad \sigma_0 \leqslant \sigma \leqslant 1 + \frac{1}{\log\log t}, \end{cases}$$

and for $\sigma > 1 - \frac{E}{\log\log t}$, $E > 0$ fixed,

$$\zeta(s) \ll e^{Le^{(2+\varepsilon)E}}(\log\log t), \tag{1}$$

where $L = L(t) = \log\log\log\log t$ and the implied constant in the $\ll$ depends on E.

For the first estimate, see [3], Chapter XIV, §14.33. The second estimate is similar to the first and is obtained along the lines of [3], Chapter XIV, §14.9. For a more precise estimate, see [4].

Lemma 2. *For $\alpha \leqslant \sigma \leqslant \beta$ and $t > 1$ we have*

$$\Gamma(\sigma + it) = t^{\sigma+it-1/2} \exp\left(-\frac{\pi}{2}t - it + i\frac{\pi}{2}\left(\sigma - \frac{1}{2}\right)\right)\sqrt{2\pi}\left(1 + O\left(\frac{1}{t}\right)\right),$$

with the constant in the big-O depending only on α and β.

For the proof, see e.g. [5], Appendix, §3.

Lemma 3. *Let $\sigma_a(n)$, $a \in \mathbb{C}$, be the sum of ath powers of the divisors of n. Let $\xi(d)$ be an arbitrary bounded arithmetical function with the support in the set of square-free integers. Then for $\sigma > 1$, $T_1, T_2 \in \mathbb{R}$ we have the identity*

$$\sum_{n=1}^{\infty} \sigma_{iT_1}(n)\sigma_{-iT_2}(n)\left(\sum_{d|n}\xi(d)\right)n^{-s}$$
$$= \frac{\zeta(s)\zeta(s-iT_1)\zeta(s+iT_2)\zeta(s-i(T_1-T_2))}{\zeta(2s-i(T_1-T_2))}\left(\xi(1) + \sum_{d=2}^{\infty}\xi(d)P_d(s,T_1,T_2)\right),$$

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where

$$P_d(s,T_1,T_2) = \prod_{p|d}\left(1-\left(1-\frac{1}{p^s}\right)\left(1-\frac{1}{p^{s-iT_1}}\right)\left(1-\frac{1}{p^{s+iT_2}}\right)\left(1-\frac{1}{p^{s-i(T_1-T_2)}}\right)\left(1-\frac{1}{p^{2s-i(T_1-T_2)}}\right)^{-1}\right).$$

Proof. This is a version of Lemma 3 of Y. Motohashi [1]. Let

$$Z=\frac{\zeta(s)\zeta(s-iT_1)\zeta(s+iT_2)\zeta(s-i(T_1-T_2))}{\zeta(2s-i(T_1-T_2))}.$$

Changing the order of summation, we have

$$\sum_{n=1}^{\infty}\sigma_{iT_1}(n)\sigma_{-iT_2}(n)\left(\sum_{d|n}\xi(d)\right)n^{-s}$$
$$=\xi(1)Z+\sum_{\substack{d\geqslant 2, d \text{ square-free}\\ d=p_{d_1}\cdots p_{d_r}}}\xi(d)\left(\sum_{k=1}^{\infty}\frac{\sigma_{iT_1}(kp_{d_1}\cdots p_{d_r})\sigma_{-iT_2}(kp_{d_1}\cdots p_{d_r})}{k^s p_{d_1}^s\cdots p_{d_r}^s}\right)$$
$$=\xi(1)Z+\sum_{d\geqslant 2, d \text{ square-free}}\xi(d)\prod_{p|d}\left(\frac{\left(1+p^{iT_1}\right)\left(1+p^{-iT_2}\right)}{p^s}+\frac{\left(1-p^{i3T_1}\right)\left(1-p^{-i3T_2}\right)}{\left(1-p^{iT_1}\right)\left(1-p^{-iT_2}\right)}\frac{1}{p^{2s}}+\dots\right)$$
$$\times\prod_{p\nmid d}\left(1+\frac{\left(1+p^{iT_1}\right)\left(1+p^{-iT_2}\right)}{p^s}+\frac{\left(1-p^{i3T_1}\right)\left(1-p^{-i3T_2}\right)}{\left(1-p^{iT_1}\right)\left(1-p^{-iT_2}\right)}\frac{1}{p^{2s}}+\dots\right)$$
$$=\xi(1)Z+\sum_{d\geqslant 2, d \text{ square-free}}\xi(d)Z\prod_{p|d}\frac{\frac{\left(1+p^{iT_1}\right)\left(1+p^{-iT_2}\right)}{p^s}+\frac{\left(1-p^{i3T_1}\right)\left(1-p^{-i3T_2}\right)}{\left(1-p^{iT_1}\right)\left(1-p^{-iT_2}\right)}\frac{1}{p^{2s}}+\dots}{1+\frac{\left(1+p^{iT_1}\right)\left(1+p^{-iT_2}\right)}{p^s}+\frac{\left(1-p^{i3T_1}\right)\left(1-p^{-i3T_2}\right)}{\left(1-p^{iT_1}\right)\left(1-p^{-iT_2}\right)}\frac{1}{p^{2s}}+\dots}.$$

By an identity of Ramanujan—Wilson [3], (1.3.3),

$$\prod_{p|d}\frac{\frac{\left(1+p^{iT_1}\right)\left(1+p^{-iT_2}\right)}{p^s}+\frac{\left(1-p^{i3T_1}\right)\left(1-p^{-i3T_2}\right)}{\left(1-p^{iT_1}\right)\left(1-p^{-iT_2}\right)}\frac{1}{p^{2s}}+\dots}{1+\frac{\left(1+p^{iT_1}\right)\left(1+p^{-iT_2}\right)}{p^s}+\frac{\left(1-p^{i3T_1}\right)\left(1-p^{-i3T_2}\right)}{\left(1-p^{iT_1}\right)\left(1-p^{-iT_2}\right)}\frac{1}{p^{2s}}+\dots}$$
$$=\prod_{p|d}\left(\frac{1-p^{i(T_1-T_2)-2s}}{\left(1-p^{-s}\right)\left(1-p^{iT_1-s}\right)\left(1-p^{-iT_2-s}\right)\left(1-p^{i(T_1-T_2)-s}\right)}-1\right)$$
$$\times\frac{\left(1-p^{-s}\right)\left(1-p^{iT_1-s}\right)\left(1-p^{-iT_2-s}\right)\left(1-p^{i(T_1-T_2)-s}\right)}{1-p^{i(T_1-T_2)-2s}}.$$

This obviously ends the proof of the lemma.

Lemma 4. *Assume the truth of the Riemann hypothesis. Fix $E>0$. Let*

$$\exp(A\log\log T\log\log\log T)\leqslant N\leqslant\exp(DA\log\log T\log\log\log T),\quad T\geqslant e^{27},$$

with $A=\frac{18+\varepsilon}{E}$ and a sufficiently large positive constant D, and let us put $T_1=T$, $T_2=T+H$,

with $H = c(\log\log T)^{-1}$. *Then we have*

$$
\begin{aligned}
&\sum_{n\leqslant N}|\sigma_{iT_1}(n)|^2|\sigma_{iT_2}(n)|^2 \\
&\qquad \ll_{A,D} N \\
&\qquad \times \big((\log\log\log T)^3(\log\log T)^7|\zeta(1+iT_1)|^4|\zeta(1+iT_2)|^4 \\
&\qquad + (\log\log T)^7\zeta(1+i(T_1+H))^2\zeta(1-i(T_1-H))^2\zeta(1+i(T_2+H))^2\zeta(1-i(T_2-H))^2 \\
&\qquad +(\log\log T)^7\zeta(1+i(T_1-H))^2\zeta(1-i(T_1+H))^2\zeta(1+i(T_2-H))^2\zeta(1-i(T_2+H))^2\big) \\
&\qquad + O\left(N(\log\log T)^{-1}\right).
\end{aligned}
$$

Proof. Let

$$F_0(s,T_1,T_2) = \sum_{n=1}^{\infty}|\sigma_{iT_1}(n)|^2|\sigma_{iT_2}(n)|^2 n^{-s} \quad (\sigma > 1).$$

By the identity of U. Balakrishnan [6], we have

$$
\begin{aligned}
F_0(s,T_1,T_2) &= \zeta(s)^4\zeta(s+iT_1)^2\zeta(s-iT_1)^2\zeta(s+iT_2)^2\zeta(s-iT_2)^2 \\
&\quad \times \zeta(s+i(T_1-T_2))\zeta(s-i(T_1-T_2))\zeta(s+i(T_1+T_2))\zeta(s-i(T_1+T_2))G(s,T_1,T_2),
\end{aligned}
$$

where $G(s,T_1,T_2)$ is regular and bounded for $\sigma \geqslant \sigma_0 > 1/2$, uniformly in T_1, T_2. The limiting case $T_1 = T_2$ gives the identity of Y. Motohashi, which is connected with the famous nonnegative trigonometric polynomial $3 + 4\cos\varphi + \cos 2\varphi$ and the inequality of Mertens. Littlewood's bound (1) and Perron's inversion formula for the height $U = N^{1+\varepsilon}$ give

$$
\begin{aligned}
\sum_{n\leqslant N}|\sigma_{iT_1}(n)|^2|\sigma_{iT_2}(n)|^2 &= \operatorname{Res}\left(F_0(s,T_1,T_2)N^s s^{-1}\right)_{s=1,1\pm iH} \\
&\quad + O\left(\left(e^{L(T)e^{(2+\varepsilon)E}}\log\log T\right)^{10}(\log\log T)^6 N^{\eta}\log U\right) \\
&= \operatorname{Res}\left(F_0(s,T_1,T_2)N^s s^{-1}\right)_{s=1,1\pm iH} + O\left(N(\log\log T)^{-1-\varepsilon}\right),
\end{aligned}
$$

where we have put

$$\eta = 1 - \frac{E}{\log\log T}.$$

Also,

$$\operatorname{Res}\left(F_0(s,T_1,T_2)N^s s^{-1}\right)_{s=1} \ll N\sum_{k=0}^{3}|(\partial s)^k_{s=1}H(s,T_1,T_2)|(\log N)^{3-k},$$

where

$$
\begin{aligned}
H(s,T_1,T_2) &= \zeta(s+iT_1)^2\zeta(s-iT_1)^2\zeta(s+iT_2)^2\zeta(s-iT_2)^2 \\
&\quad \times \zeta(s+i(T_1-T_2))\zeta(s-i(T_1-T_2))\zeta(s+i(T_1+T_2))\zeta(s-i(T_1+T_2)).
\end{aligned}
$$

By taking the logarithmic derivative, we get

$$(\partial s)^k_{s=1}H(s,T_1,T_2) \ll H(1,T_1,T_2)(\log\log T\log\log\log T)^k.$$

From the theorem of Littlewood and the definition of H we see that

$$\zeta(1+i(T_1-T_2))\zeta(1-i(T_1-T_2))\zeta(1+i(T_1+T_2))\zeta(1-i(T_1+T_2)) \ll (\log\log T)^4,$$

which implies the assertion of the lemma.

Lemma 5. *Let $\mu(d)$ be the Möbius function, and let*

$$\lambda_d(z) = \begin{cases} \mu(d) & \text{if} \quad d < z, \\ \mu(d)\frac{\log\left(z^2/d\right)}{\log z} & \text{if} \quad z \leqslant d < z^2, \\ 0 & \text{otherwise}, \end{cases}$$

where $z > 1$ is arbitrary. Then we have, uniformly in $N > 1$ and in z,

$$\sum_{n \leqslant N} \left(\sum_{d|n} \lambda_d(z) \right)^2 \ll \frac{N}{\log z}.$$

This lemma is due to Barban—Vehov [7] and appears as Lemma 5 in Y. Motohashi [1]. For the proof, see [8] and [9].

Lemma 6. *For any large y, and fixed a, $q > 1$, $(a, q) = 1$,*

$$\sum_{\substack{p \leqslant y \\ p \equiv a \pmod q}} \operatorname{sgn}\left(\cos(2h \log p)\right) \frac{\cos(h \log p)}{p} = \frac{1}{\varphi(q)} \log\left(\min\left(h^{-1}, \log y\right)\right) + O(1) \quad \text{for } 0 < h < c.$$

This and related estimates can be proved by using PNT in arithmetic progressions and Stieltjes integration. A similar lemma can be found in [10].

3. Proof of Theorem. We put

$$X = \exp(0.5DA \log\log T \log\log\log T), \quad z = \exp(A \log\log T \log\log\log T) \tag{2}$$

with the same A and D as in Lemma 4, set $\xi(d) = \lambda_d(z)$ in Lemma 3 and for $T_1 = T$, $T_2 = T+H$ with $H = c(\log\log T)^{-1}$, write

$$J(s, T_1, T_2) = \frac{\zeta(s)\zeta(s - iT_1)\zeta(s + iT_2)\zeta(s - i(T_1 - T_2))}{\zeta(2s - i(T_1 - T_2))},$$

$$K(s, T_1, T_2) = \sum_{d \leqslant z^2} \lambda_d(z) P_d(s, T_1, T_2).$$

Theorem 1. *Assume the Riemann hypothesis. Then there exists an infinite sequence of pairs of real numbers (T_1, T_2), $T_1 = T$, $T_2 = T + H$, with arbitrarily large values of T and $H = c(\log\log T)^{-1}$, such that*

$$|\zeta(1 + iT_1)||\zeta(1 + iT_2)| \ll (\log\log T)^{-2}$$

and

$$\begin{aligned} &(\log\log T)^7 |\zeta(1 + iT_1)|^4 |\zeta(1 + iT_2)|^4 \\ &+(\log\log T)^7 \zeta(1 + i(T_1 + H))^2 \zeta(1 - i(T_1 - H))^2 \zeta(1 + i(T_2 + H))^2 \zeta(1 - i(T_2 - H))^2 \\ &+(\log\log T)^7 \zeta(1 + i(T_1 - H))^2 \zeta(1 - i(T_1 + H))^2 \zeta(1 + i(T_2 - H))^2 \zeta(1 - i(T_2 + H))^2 \\ &\ll (\log\log T)^{-1}. \end{aligned}$$

Let $s_0 = \sigma_0 + it_0$ be a point such that

$$|J(s_0, T_1, T_2) K(s_0, T_1, T_2)| \geqslant (\log\log T)^{\varepsilon} \tag{3}$$

with arbitrarily small fixed $\varepsilon > 0$, and

$$\sigma_0 = 1 - \frac{E_0}{\log\log T} \geqslant 1 - \frac{E}{\log\log T}, \quad C \log\log\log T \leqslant |t_0| \leqslant T/2. \tag{4}$$

Then $E_0 \geqslant c_2(\varepsilon) > 0$.

Proof. By Mellin's inversion formula, when $c - \sigma_0 > 0$,

$$e^{-n/X} = \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \Gamma(s - s_0) \frac{X^{s-s_0}}{n^{s-s_0}}\, ds.$$

Hence for $c > 1$ and $c > \sigma_0$ by Lemma 3 we have that

$$\begin{aligned} e^{-1/X} &+ \sum_{n \geqslant z} \sigma_{iT_1}(n)\sigma_{-iT_2}(n) n^{-s_0} a(n) e^{-n/X} \\ &= \frac{X^{-s_0}}{2\pi i} \int\limits_{(\sigma=c)} J(s, T_1, T_2) K(s, T_1, T_2) \Gamma(s - s_0) X^s\, ds, \end{aligned}$$

where

$$a(n) = \sum_{d|n} \lambda_d(z).$$

We now move the line of integration to the line

$$\sigma = \eta = 1 - \frac{E}{\log\log T}.$$

There are simple poles at $s = 1,\ 1 + iT_1,\ 1 - iT_2,\ 1 + i(T_1 - T_2)$, but by (4) and Lemma 2 they leave residues that are all bounded by $O\left((\log\log T)^{-2}\right)$. Now we consider the estimation of the integral along $\sigma = \eta$. For the estimation of $K(s, T_1, T_2)$ we define the generating Dirichlet series

$$\begin{aligned} M_w(s, T_1, T_2) &= 1 + \sum_{d=2}^{\infty} \mu(d) P_d(s, T_1, T_2) d^{-w} \\ &= \prod_p \left(1 - \frac{1}{p^w}\left(1 - \left(1 - \frac{1}{p^s}\right)\left(1 - \frac{1}{p^{s-iT_1}}\right)\left(1 - \frac{1}{p^{s+iT_2}}\right)\left(1 - \frac{1}{p^{s-i(T_1-T_2)}}\right)\right.\right. \\ &\quad \left.\left. \times \left(1 - \frac{1}{p^{2s-i(T_1-T_2)}}\right)^{-1}\right)\right). \end{aligned}$$

Using a version of Perron's inversion formula, we get

$$\frac{1}{1!} \sum_{d \leqslant z^2} \mu(d) P_d(s, T_1, T_2) \log\left(z^2/d\right) = \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} M_w(s, T_1, T_2) \frac{z^{2w}}{w^2}\, dw,$$

with $c = 1 - \Re s + \frac{1}{\log z}$, which implies that on the line $\Re s (= \sigma) = \eta$ we have

$$K(s, T_1, T_2) \ll z^{2(1-\eta)} (\log z)^{10} \ll \exp(2AE \log\log\log T)(\log\log T \log\log\log T)^{10}.$$

Thus recalling (4), (3) and (2) we get, as in the proof of Lemma 4, that

$$\begin{aligned} &\Bigg| \operatorname{Res}\left(X^{-s_0} J(s, T_1, T_2) K(s, T_1, T_2) \Gamma(s - s_0) X^s\right)_{s=s_0} \\ &+ \frac{X^{-s_0}}{2\pi i} \int\limits_{(\sigma=\eta)} J(s, T_1, T_2) K(s, T_1, T_2) \Gamma(s - s_0) X^s ds - e^{-1/X} \Bigg| \\ &\geqslant (\log\log T)^{\varepsilon} + O\left(\exp\left(0.5DA \log\log\log T (E_0 - E)\right) \frac{\log\log T}{E - E_0}\right. \\ &\quad \left. \times \left(e^{L(T) e^{(2+\varepsilon)E}} \log\log T\right)^4 (\log\log T)^{2AE+10+\varepsilon}\right). \end{aligned}$$

Hence there is an N such that $z \leqslant N \leqslant X^2$, and

$$\sum_{N \leqslant n \leqslant 2N} |\sigma_{iT_1}(n)||\sigma_{-iT_2}(n)||a(n)|n^{-\sigma_0} \gg (\log\log T)^{-1+\varepsilon},$$

since the range of the summation $z \leqslant n \leqslant X^2$ may be divided into the intervals $N \leqslant n \leqslant 2N$ so that the number of the intervals is $\ll \log X^2/z \ll \log\log T \log\log\log T$ and the sum over the entire range must be $\gg (\log\log T)^{\varepsilon}$. By the Cauchy inequality and by Lemma 5, we get

$$(\log\log T)^{-2+\varepsilon} \log z \ll \sum_{N \leqslant n \leqslant 2N} |\sigma_{iT_1}(n)|^2 |\sigma_{iT_2}(n)|^2 N^{1-2\sigma_0}.$$

Finally, by Lemma 4 with $T_1 = T$, $T_2 = T + H$ we establish that

$$\begin{aligned} N^{2(1-\sigma_0)} \gg \big(& (\log\log\log T)^3 (\log\log T)^7 |\zeta(1+iT_1)|^4 |\zeta(1+iT_2)|^4 \\ & + (\log\log T)^7 \zeta(1+i(T_1+H))^2 \zeta(1-i(T_1-H))^2 \zeta(1+i(T_2+H))^2 \zeta(1-i(T_2-H))^2 \\ & + (\log\log T)^7 \zeta(1+i(T_1-H))^2 \zeta(1-i(T_1+H))^2 \zeta(1+i(T_2-H))^2 \zeta(1-i(T_2+H))^2 \big) \\ & + O\left((\log\log T)^{-1}\right)\big)^{-1} (\log\log T)^{-1+\varepsilon}. \end{aligned}$$

Next we prove existence of the infinite sequence of pairs of real numbers (T_1, T_2), claimed in the theorem. We may choose $T_1 = T$ and $T_2 = T + H$ in the following way: As in [3], Chapter VIII, §8.6, for $\sigma > 1$

$$\log \frac{1}{|\zeta(s)|} = -\sum \frac{\cos(t \log p_n)}{p_n^{\sigma}} + O(1).$$

Also, we have the identity

$$\cos((t+h)\log p_n) = \cos(t \log p_n)\cos(h \log p_n) - \sin(t \log p_n)\sin(h \log p_n).$$

So, we want to choose t such that for, say, every $p_n \equiv \pm 1 \pmod 7$ and $n \leqslant N_2$

$$\cos(t \log p_n) < -1 + \frac{1}{N_2},$$

for every $p_n \equiv \pm 2 \pmod 7$ and $n \leqslant N_2$

$$\cos(t \log p_n) \begin{cases} < -1 + \frac{1}{N_2} & \text{if} \quad \cos(H \log p_n) \geqslant 0, \\ > 1 - \frac{1}{N_2} & \text{if} \quad \cos(H \log p_n) < 0, \end{cases}$$

and for every $p_n \equiv \pm 3 \pmod 7$ and $n \leqslant N_2$

$$\cos(t \log p_n) \begin{cases} < -1 + \frac{1}{N_2} & \text{if} \quad \cos(2H \log p_n) \geqslant 0, \\ > 1 - \frac{1}{N_2} & \text{if} \quad \cos(2H \log p_n) < 0. \end{cases}$$

This may be done as in Lemma δ of [3], Chapter VIII, §8.8. Now existence of the sequence (T_1, T_2) follows from this and estimates as in Lemma 6 by the Phragmén–Lindelöf method. Thus,

$$\begin{aligned} N^{2(1-\sigma_0)} \gg & \left((\log\log\log T)^3 (\log\log T)^{-1} + O\left((\log\log T)^{-1}\right)\right)^{-1} \\ & \times (\log\log T)^{-1+\varepsilon}. \end{aligned}$$

This ends the proof of the theorem.

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